ASCHAM SCHOOL
2002 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 1
FORM VI

General Instructions:
- Reading Time: 5 minutes
- Working Time: 2 hours
- Write using blue or black pen
- Approved calculators and templates may be used.
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question.

Collection:
- Start each question in a new answer book
- Write your name and teacher's name on each book.
- If you use a second book, place it inside the first.

Total Marks : 84
- Attempt Questions 1 – 7
- All questions are of equal value.

Question 1 ***Start a new answer book***

a) Express $\frac{5\pi}{12}$ radians as degrees [1]

b) Find a primitive of e^{-2x} [1]

c) Use the table of standard integrals to find the exact value of

$$\int_0^4 \frac{dx}{\sqrt{x^2+4}}$$ [2]

d) If α, β and γ are roots of the equation $6x^3+7x^2-x-2=0$, find the value of

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$$ [3]

e) Find the domain and range of $f(x)=4\sin^{-1}\frac{x}{3}$, and sketch the graph of *f(x)* .[3]

f) Find $\frac{d}{dx}e^{\cos x}$ [2]

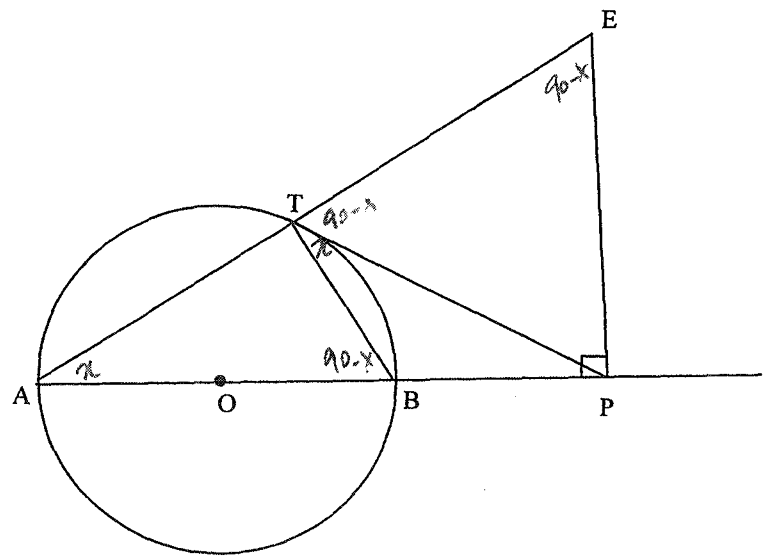
Question 2 ***Start an new answer book***

a) Find (i) $\int \frac{x}{4+x^2}\,dx$ [1]

(ii) $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \tan^2 2x\,dx$ [3]

b) If $\sum_{k=4}^{\infty} 2r^{k-3}=10$, find r if r exists. [3]

c)



Make a large neat copy of the diagram in your answer book.

AB is the diameter of the circle with centre O. TP is a tangent to the circle at T. EP⊥AP. Prove:

(i) TBPE is a cyclic quadrilateral [2]
(ii) PT = PE. [3]

Question 3 ***Start a new answer book***

a) Solve : $\dfrac{2x}{5-x} \geq 1$ [3]

b) Prove by mathematical induction that $6^n - 1$ is divisible by 5 for all positive integers.

[5]

c) By substituting $t = \tan\frac{x}{2}$, find the solutions to the equation:

$3\sin x + 4\cos x = 5$ for $0^\circ \leq x \leq 360^\circ$,

giving your answers correct to the nearest degree.
[4]

Question 4 **Start a new answer book**

a) Using the substitution $u = x^3 + 1$, evaluate $\int_{-1}^{1} x^2(x^3 + 1)\,dx$ [2]

b) (i) Factorise : $x^3 - 3x + 2$
(ii) Hence draw a neat sketch of the polynomial $y = x^3 - 3x + 2$ without the use of calculus, showing all intercepts with the co-ordinate axes.
(iii) Hence solve the inequality $x^3 - 3x + 2 > 0$ [4]

c) Find the value of $\sin\left(2\sin^{-1}\frac{2}{3}\right)$ in exact form
[3]

d) i) Show that the equation $f(x) = x^3 - 8x + 8$ has a zero between –3 and –4 .
ii) Taking $x = -3 \cdot 5$ as a first approximation of the solution of the equation $f(x) = 0$, use Newton's method once to find a closer approximation, giving your answer to 2 decimal places
[3]

Question 5 **Start a new answer book**

a) A bug is oscillating in simple harmonic motion such that its displacement x metres from a fixed point O at time t seconds is given by the equation $\ddot{x} = -4x$. When $t = 0$, $v = 2$ m/s and $x = 5$.
(i) Show that $x = a\cos(2t + \beta)$ is a solution for this equation, where a and β are constants.
(ii) Find the period of the motion.
(iii) Show that the amplitude of the oscillation is $\sqrt{26}$.
(iv) What is the maximum speed of the bug? [5]

b) (i) Prove that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$ [2]
(ii) The acceleration of a creature is given by $\ddot{x} = -\frac{1}{2}u^2e^{-x}$ where x is the displacement from the origin, and u is the initial velocity at the origin. Given that $u = 2$ m/s:
(α) Show that $v^2 = 4e^{-x}$
(β) Explain why $v > 0$, and find x in terms of t.
(γ) Describe the subsequent motion of the creature as $t \to \infty$.
[5]

Question 6 **Start a new answer book**

a) A ladder is slipping down a vertical wall. The ladder is 4 metres long. The top of the ladder is slipping down at a rate of 3 m/s. How fast is the bottom of the ladder moving along the ground when the bottom is 2 metres away from the foot of the wall? [4]

b) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The focus S is the point $(0, a)$. The tangent at P meets the y-axis at Q.

(i) Find the equation of the tangent at P and the co-ordinates of Q.
(ii) Prove that $SP = SQ$
(iii) Hence show that $\angle PSQ + 2\angle SQP = 180^\circ$ [4]

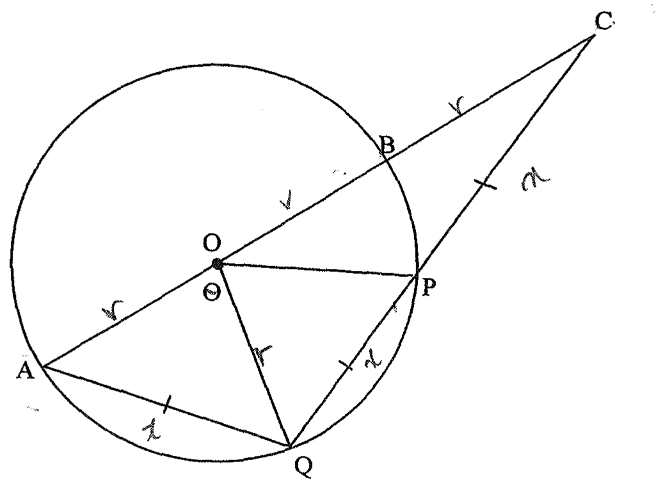
c) In a town in Mathsland, a 'flu epidemic is spreading at a rate proportional to the population that have it, such that is it predicted that the number of people who have the disease will double in 3 weeks, i.e. $\frac{dA}{dt} = kA$, where A is the number of people with 'flu in time t weeks.

(i) Show that $A = A_0 e^{kt}$, where A_0 is the initial number of people with 'flu, satisfies the above differential equation.

(ii) Find k in exact form

(iii) In the neighbouring town with a population of 20,000, three people have the 'flu. How many weeks (to the nearest week) will it take for the whole population to contract the disease?
[4]

Question 7 Start a new book



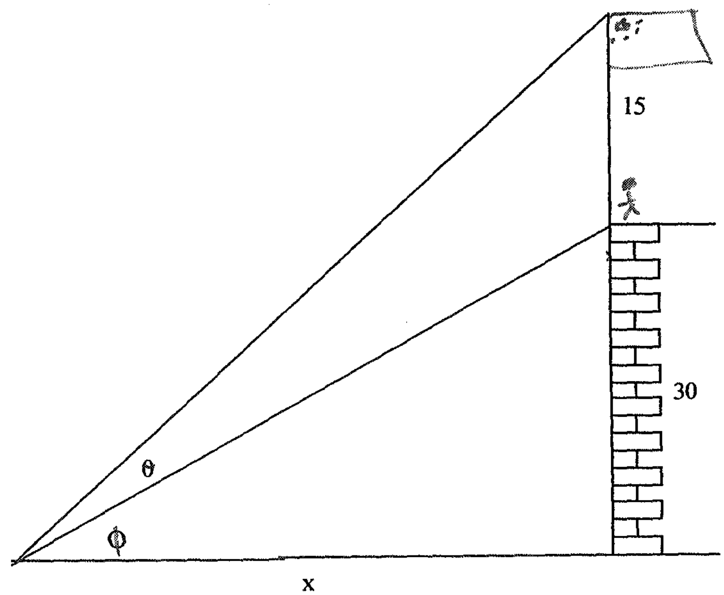
a) *Make a large neat copy of the diagram in your answer book.*

AB is the diameter of the circle with centre O and radius r. BC = r, AQ = QP =PC, and $\angle AOQ = \theta$

(i) Prove that $\cos\theta = \frac{1}{4}$ (Hint: use the cosine rule in triangles AQO and QOC) [5]

(ii) Hence prove that QC = $r\sqrt{6}$ [2]

b) A 15 metre high flagpole stands on top of a building which is 30 m high. The flagpole subtends an angle of θ degrees to a point x metres from the foot of the building, and the building subtends an angle of ϕ degrees to the



same point.

i) Show that $\theta = \tan^{-1}\left(\dfrac{15x}{x^2+1350}\right)$

ii) Hence find the value of x which will make θ a maximum. [6]

End of Examination

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

ASCHAM

SOLUTIONS : EXTENSION 1 TRIAL EXAM 2002

DKG

1. a) $\frac{5\pi}{12} = \boxed{75^\circ}$

b) $\int e^{-2x}\,dx = \boxed{-\frac{1}{2}e^{-2x}+c}$

c) $\int_0^4 \frac{dx}{\sqrt{x^2+4}}\,dx = \left[\log_e\left(x+\sqrt{x^2+4}\right)\right]_0^4$

$= \log_e(4+\sqrt{20}) - \log_e(0+2)$

$= \log_e(4+2\sqrt{5}) - \log_e 2$

$= \boxed{\log_e(2+\sqrt{5})}$ //

d) $6x^3+7x^2-x-2=0$

$\alpha\beta\gamma = -\frac{d}{a} = \frac{1}{3}$

$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = -\frac{1}{6}$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$= \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{-\frac{1}{6}}{\frac{1}{3}}$

$= \boxed{-\frac{1}{2}}$ //

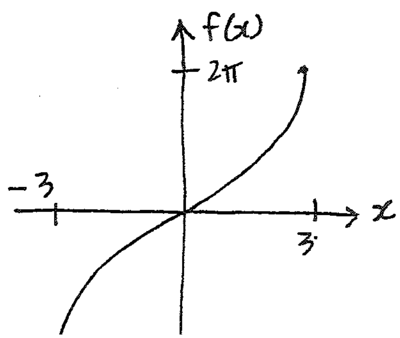
e) $f(x) = 4\sin^{-1}\frac{x}{3}$

let $y = 4\sin^{-1}\frac{x}{3}$

$\therefore \frac{y}{4} = \sin^{-1}\frac{x}{3}$

range: $-2\pi \le y \le 2\pi$

dom: $-1 \le \frac{x}{3} \le 1$



2 a) i) $\int \frac{x}{4+x^2}\,dx = \frac{1}{2}\int\frac{2x}{4+x^2}\,dx = \boxed{\frac{1}{2}\log(4}$

ii) $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \tan^2 2x\,dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{6}}(\sec^2 2x - 1)\,dx$

$= \left[\frac{1}{2}\tan 2x - x\right]_{\frac{\pi}{8}}^{\frac{\pi}{6}}$

$= \frac{1}{2}\tan\frac{\pi}{3} - \frac{\pi}{6} - \frac{1}{2}\tan\frac{\pi}{4}$ ·

$= \frac{1}{2}\sqrt{3} - \frac{\pi}{6} - \frac{1}{2} + \frac{\pi}{8}$

$= \frac{1}{2}\sqrt{3} - \frac{1}{2} - \frac{\pi}{24}$

$= \boxed{\frac{1}{24}(12\sqrt{3}-12-\pi)}$

f) $\frac{d}{dx}e^{\cos x} = \boxed{-\sin x . e^{\cos x}}$

b) $\sum_{k=4}^{\infty} 2r^{k-3} = 2r^1 + 2r^2 + 2r^3 + \ldots$

so $10 = 2(r+r^2+r^3\ldots+)$ -

$\therefore 5 = r+r^2+r^3+\ldots$

$\therefore 5 = \frac{r}{1-r}$

$a=r \quad r=r \quad \therefore$

$S_\infty = \frac{a}{1-r}$

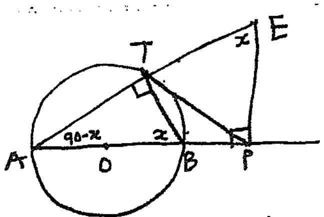
$\therefore 5-5r = r$

$6r = 5$

$r = \frac{5}{6}$

for S_∞ $\quad -1 < r < r$

$\therefore$ $\boxed{r = \frac{5}{6}}$



(i) $\angle ATB = 90$ ($\angle$ in semi ⊙)

$= \angle EPB$

$\therefore$ TBPE is cyclic QED.

(ext $\angle$ = int opp $\angle$ of quad).

(ii) let $\angle E = x$

$\therefore \angle TAB = 90 - x$ ($\angle$'s of ΔEPA)

$\therefore \angle PTB = 90 - x$ ($\angle$ in alt seg.)

$\therefore \angle ETP = x$ ($\angle$'s on str line)

$\therefore TP = TE$ QED. (sides opp = $\angle$'s)

$\frac{2x}{5-x} \geq 1$ $\qquad x \neq 5$

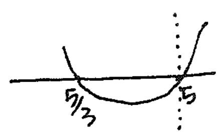
$\boxed{\times (5-x)^2}$

$2x(5-x) \geq (5-x)^2$

$10x - 2x^2 \geq 25 - 10x + x^2$

$3x^2 - 20x + 25 \leq 0$

$(3x-5)(x-5) \leq 0$



Sol^n: $\frac{5}{3} \leq x < 5$ //

let $P(n) = 6^n - 1$

3b) when $n=1$ $\quad 6^n - 1 = 6 - 1$

$= 5$

which is divisible by 5

let us assume $\exists k$ such that

$6^k - 1 = 5m$ for some $m \in \mathbb{Z}$

we want to show that $P(k+1)$ is divis by 5

i.e. $6^{k+1} - 1$ is divisible by 5.

now $6^{k+1} - 1 = 6^k . 6 - 1$

from ①: $6^k = 5m + 1$

so $6^{k+1} - 1 = (5m+1)6 - 1$

$= 6.5m + 6 - 1$

$= 6.5m + 5$

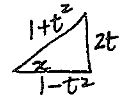
$= 5(6m + 1)$

which is divisible by 5.

so by the process of mathematical induction, the statement $P(n)$ is true for all $n \in \mathbb{Z}^+$

$3\sin x + 4\cos x = 5$

let $t = \tan\frac{1}{2}x$



$$\frac{6t}{1+t^2} + \frac{4(1-t^2)}{1-t^2} = 5$$

$6t + 4 + 4t^2 = 5 + 5t^2$

$9t^2 - 6t + 1 = 0$

$(3t-1)(3t-1) = 0$

$\therefore t = \frac{1}{3}$ for $0° \le x \le 180°$

so $\tan\frac{1}{2}x = \frac{1}{3}$

$\frac{1}{2}x = 18°26$

$x = 36°52'$

testing $x = 180°$:

LHS $= 3 \times 0 + 4 \times -1$

$= -4$

$\neq$ RHS

so soln is $x = 36°52' = 37°$ (to n degree) #

] $\int_{-1}^{1} x^2 (x^3+1)\,dx$

$u = x^3 + 1$

$du = 3x^2\,dx$

$\therefore x^2\,dx = \frac{1}{3}\,du$

when $x = 1$, $u = 2$

$x = -1$, $u = 0$

$= \frac{1}{3}\int_0^2 u\,du$

$= \frac{1}{3}\left[\frac{1}{2}u^2\right]_0^2$

$= \frac{1}{6}(4 - 0)$

$= \boxed{\frac{2}{3}}$ #

4 b) i) $f(x) = x^3 - 3x + 2$

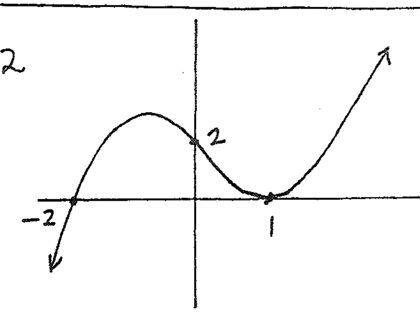
$f(1) = 1 - 3 + 2 = 0$

so $(x-1)$ is a factor

$$\begin{array}{r} x^2 + x - 2 \\ x-1 \overline{)\, x^3 \qquad - 3x + 2} \\ \underline{x^3 - x^2} \qquad\qquad \\ x^2 - 3x \qquad \\ \underline{x^2 - x} \qquad \\ -2x + 2 \\ -2x + 2 \end{array}$$

so $f(x) = (x-1)(x+2)(x-1)$ #

ii) y-int = 2



iii) $x^3 - 3x + 2 > 0$

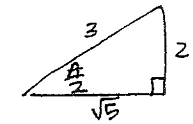
for $-2 < x < 1$ and $x > 1$ (or $x >$ #

c) $\sin(2\sin^{-1} 2/3)$

let $A = \sin^{-1} 2/3$

$\therefore A = \sin^{-1} 2/3$

$\therefore \sin A = 2/3$



now $\sin 2A = 2\sin A \cos A$

1 i) $f(x) = x^3 - 8x + 8$

$f(-3) = (-3)^3 - 8(-3) + 8$

$= 5$

$f(-4) = (-4)^3 - 8(-4) + 8$

$= -24$

$\therefore f(x)$ has a zero b/w -3 and -4

i)

$x_1 = x_0 - \frac{f(x_0)}{f'(x)_0}$ $\qquad f'(x) = 3x^2 - 8$

$\therefore x_1 = -3.5 - \frac{(-3.5)^3 - 8(3.5) + 8}{3(-3.5)^2 - 8}$

$= -3.26086\ldots$

$\doteqdot -3.26$

so $x = -3.26$ is an approx for $f(x) = 0$.

7

1) i) $x = a\cos(2t + \beta)$

$\therefore v = -2a\sin(2t + \beta)$

$\ddot{x} = -4a\cos(2t + \beta)$

so $\ddot{x} = -4x$ and satisfies eq^n. #

ii) $T = \frac{2\pi}{n}$

$T = \frac{2\pi}{2}$

$= \pi$ #

iii) when $t = 0$, $x = 5$

$\therefore 5 = a\cos\beta$ — [I]

when $t = 0$, $v =$

$\therefore 2 = -2a\sin$

$-1 = a\sin\beta$

squaring and adding:

$26 = a^2$

$\therefore a = \sqrt{26}$ since $a > 0$

so amp $= \sqrt{26}$ #

iv) $v = -2\sqrt{26}\sin(2t + \beta)$

so max speed is $2\sqrt{26}$ m/s #

(since $-2\sqrt{26}\sin(2t+\beta)$ has max

5 b) i) to prove: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

proof: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \frac{d^2x}{dt^2}$

$= \ddot{x}$ QED.

ii) α) $\ddot{x} = -\frac{1}{2}u^2 e^{-x}$

$u = 2$ m/s

$\therefore \ddot{x} = -\frac{1}{2} \times 4 e^{-x}$

$\ddot{x} = -2e^{-x}$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\therefore \frac{1}{2}v^2 = 2e^{-x} + \frac{1}{2}c_1$

$\therefore v^2 = 4e^{-x} + c_1$

when $x = 0$, $v = 2$

$\therefore 4 = 4 + c_1$

so $c_1 = 0$

$\therefore v^2 = 4e^{-x}$ Q.E.D.

β) $4e^{-x} > 0$ for all x

so v^2 does not change sign.

since $v = 2$ at $x = 0$, it remains positive.

$\therefore \boxed{v = 2e^{-\frac{x}{2}}}$

$\therefore \frac{dx}{dt} = \frac{2}{e^{x/2}}$

$\therefore \frac{dt}{dx} = \frac{e^{x/2}}{2}$

$\therefore t = \frac{1}{2} \cdot 2e^{x/2} + c_2$

$t = e^{\frac{x}{2}} + c_2$

at $x = 0$, $t = 0$

$0 = 1 + c_2$

$\therefore c_2 = -1$

$\therefore t = e^{x/2} - 1$

$\therefore e^{x/2} = t + 1$

$\log(t+1) = \frac{x}{2}$

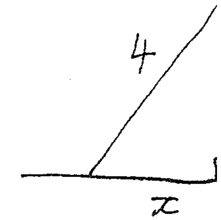
$\therefore \boxed{x = 2\log_e(t+1)}$

γ) as $t \to \infty$, $x \to \infty$] so bug displacement increases ... it

$v \to 0$

6 a) $\frac{dy}{dt} = -3$ m/s

we want $\frac{dx}{dt}$ when $x = 2$.



by pythag:

$x^2 = 16 - y^2$

$\therefore x = \sqrt{16 - y^2} = (16 - y^2)^{\frac{1}{2}}$

$\therefore \frac{dx}{dy} = \frac{1}{2}(16 - y^2)^{-\frac{1}{2}} \cdot -2y$

$\frac{dx}{dy} = \frac{-y}{\sqrt{16 - y^2}}$

also, when $x = 2$, $y^2 = 16 - 4$

$= 12$

$\therefore y = 2\sqrt{3}$

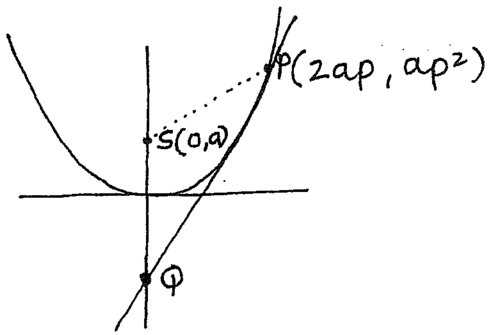
now $\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$

$= \frac{-y}{\sqrt{16 - y^2}} \times -3$

$= \frac{-2\sqrt{3}}{\sqrt{16 - 12}} \times -3$

$= 3\sqrt{3}$

so foot is sliding away at rate of $3\sqrt{3}$ m/s



). grad of tang $= p$

$\therefore$ eqⁿ is $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

when $x = 0$, $y = -ap^2$

$\therefore\ Q = (0, -ap^2)$ //

i) $\text{dist}^2 SQ = (a + ap^2)^2$

$\text{dist}^2 SP = (2ap)^2 + (ap^2 - a)^2$

$= 4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2$

$= a^2 + 2a^2p^2 + a^2p^4$

$= (a + ap^2)^2$

$\therefore$ SP = SQ QED. //

) $\angle SQP = \angle SPQ$ (SP = SQ)

$\therefore \angle PSQ + 2\angle SQP = 180°$ ($\angle$ sum of $\triangle PSQ$) //

6 c) i) sub $A = A_0 e^{kt}$ into $\frac{dA}{dt} = kA$

$LHS = \frac{d}{dt}(A_0 e^{kt})$

$= A_0 . k e^{kt}$

$RHS = k(A_0 e^{kt})$

$= A_0 k e^{kt}$

$= LHS$ Q.E.D. //

ii) when $t = 3$, $A = 2A_0$

$\therefore\ 2A_0 = A_0 e^{3k}$

$e^{3k} = 2$

$k = \frac{1}{3}\log_e 2$

iii) $A = A_0 e^{kt}$ where $k = \frac{1}{3}\ln 2$

when $A_0 = 3$, $A = 20,000$

$\therefore\ 20,000 = 3e^{kt}$

$e^{kt} = \frac{20000}{3}$

$\therefore\ t = \frac{1}{k}\log_e \frac{20000}{3}$

$= 38.108249\ldots$ weeks

$\therefore\ t \doteqdot 39$

it will take 39 complete weeks //

let $AQ = x$

i) in $\triangle AQO$

$$x^2 = 2r^2 - 2r^2\cos\theta \quad \text{—} \boxed{1}$$

in $\triangle QOC$

$$QC^2 = r^2 + 4r^2 - 2r.2r\cos(180-\theta)$$

$$4x^2 = 5r^2 + 4r^2\cos\theta$$

$$\therefore x^2 = \frac{5}{4}r^2 + r^2\cos\theta \quad \text{—} \boxed{2}$$

so $2r^2 - 2r^2\cos\theta = \frac{4}{5}r^2 + r^2\cos\theta$

$$2 - 2\cos\theta = \frac{5}{4} + \cos\theta$$

$$3\cos\theta = \frac{3}{4}$$

$$\cos\theta = \frac{1}{4} \qquad \text{QED.}$$

ii) from $\boxed{2}$

$$QC^2 = 5r^2 + 4r^2\cos\theta$$

$$QC^2 = 5r^2 + 4r^2 \times \frac{1}{4}$$

$$QC^2 = 6r^2$$

$$QC = r\sqrt{6} \qquad \text{Q.E.D.}$$

7b) i) $\tan\phi = \frac{30}{x}$

$$\tan(\theta+\phi) = \frac{45}{x}$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta.\tan\phi}$$

$$\frac{45}{x} = \frac{\tan\theta + \frac{30}{x}}{1 - \tan\theta.\frac{30}{x}}$$

$$\frac{45}{x} = \frac{x\tan\theta + 30}{x - 30\tan\theta}$$

$$\therefore 45x - 1350\tan\theta = x^2\tan\theta + 30x$$

$$(x^2 + 1350)\tan\theta = 45x - 30x$$

$$\tan\theta = \frac{15x}{x^2+1350}$$

$$\theta = \tan^{-1}\left(\frac{15x}{x^2+1350}\right).$$

ii) $$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{15x}{x^2+1350}\right)^2} \times \frac{(x^2+1350)15 - 15x.2x}{(x^2+1350)^2}$$

$$= \frac{15x^2 + 20250 - 30x^2}{(x^2+1350)^2 + (15x)^2}$$

$$= \frac{20250 - 15x^2}{(x^2+1350)^2 + (15x)^2}$$

max occurs when $\frac{d\theta}{dx} = 0$

$$\therefore 15x^2 = 20250$$

$$x^2 = 1350$$

$$x = 15\sqrt{6}$$

test:

x	36	$15\sqrt{6}$ (36.7)
$\frac{d\theta}{dx}$	810 +	0